$alpha$-Modulation Spaces and the Cauchy Problem for Nonlinear Schrodinger Equations (Harmonic Analysis and Nonlinear Partial Differential Equations)

HAN, JINSHENG; WANG, BAOXIANG

RIMS Kôkyûroku Bessatsu
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By

Jinsheng Han* and Baoxiang Wang**

Abstract

The α-modulation spaces, introduced by P. Gröbner [11], are proposed as intermediate function spaces between modulation space and Besov space. In this paper we survey our recent works on some properties of α-modulation spaces including its dual spaces, complex interpolation, algebra structures, scaling property and the embedding between different α-modulation spaces and Besov spaces. We then outline our recent results on the initial value problem of nonlinear Schrödinger equations in α-modulation spaces. Our results contain the global in time solutions with small initial data in α-modulation spaces $M_{2,1}^{s,\alpha}$, which can be out of the control of critical Sobolev spaces H^{s_c}.

§ 1. Backgrounds

It is well known that Besov space can be constructed via dyadic decomposition to frequency space. Let $\{\varphi_j\}_{j=0}^{\infty}$ be a sequence of smooth functions with φ_0 supported in the unit ball $\mathcal{B}(0,1)$ and φ_j $(j \geq 1)$ supported in the dyadic $\mathcal{C}_j = \{\xi \in \mathbb{R}^n : 2^{j-2} \leq |\xi| \leq 2^{j+2}\}$ satisfying

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n.$$

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Supported by National Science Foundation of China, grants 11271023, 11026053
*Department of Mathematics, Shanghai Jiaotong University, Shanghai 200240, China.
e-mail: pkuhan@sjtu.edu.cn
**LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China.
e-mail: wbx@math.pku.edu.cn

Let $\mathfrak{S}(\mathbb{R}^n)$ and $\mathfrak{S}'(\mathbb{R}^n)$ be the Schwartz space and its dual space, respectively. For $0 < p, q \le \infty$ and $s \in \mathbb{R}$, we can define Besov space as

$$B^s_{p,q}(\mathbb{R}^n) := \left\{ f \in \mathfrak{S}'(\mathbb{R}^n) : \|f\|_{B^s_{p,q}} := \left(\sum_{j=0}^{\infty} 2^{jsq} \|\varphi_j(D) f\|^q_{L^p} \right)^{\frac{1}{q}} < \infty \right\},$$

with some usual modification when $q = \infty$.

Modulation spaces which were introduced by Feichtinger [10] in 1983 can be defined via the frequency-uniform decompositions. Let $\{\sigma_k\}_{k\in\mathbb{Z}^n}$ be a sequence of smooth functions with σ_k supported in $k + [-1,1]^n$ satisfying

$$\sum_{k\in\mathbb{Z}^n} \sigma_k(\xi) = 1, \quad \forall\, \xi \in \mathbb{R}^n.$$

For $0 < p, q \le \infty$ and $s \in \mathbb{R}$, modulation spaces are defined as

$$M^s_{p,q}(\mathbb{R}^n) := \left\{ f \in \mathfrak{S}'(\mathbb{R}^n) : \|f\|_{M^s_{p,q}} := \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq} \|\sigma_k(D) f\|^q_{L^p} \right)^{\frac{1}{q}} < \infty \right\},$$

with some usual modification when $q = \infty$.

One easily sees that the essential difference between these two types of decompositions is that the diameters of supp φ_j and supp σ_k are $O(2^j)$ and $O(1)$, respectively.

The α-modulation spaces $M^{s,\alpha}_{p,q}$, introduced by Gröbner [11], are proposed to be intermediate function spaces to connect modulation space and Besov space, with respect to parameters $\alpha \in [0,1]$, using the concepts of α-covering to frequency space and the corresponding p-BAPU. A countable set $\mathcal{Q}$ of subsets $Q \in \mathbb{R}^n$ is called an α-covering provided that[1]

$$\bigcup_{Q\in\mathcal{Q}} Q = \mathbb{R}^n, \quad |Q| \sim \langle \xi_0\rangle^{n\alpha}, \ \forall\, \xi_0 \in Q,$$

and for any Q, there exists at most finitely many $Q' \in \mathcal{Q}$ which intersects Q and the number of such Q' has a finite upper bound which is independent of Q. This forms a third decomposition to $\mathbb{R}^n$, where we want to emphasize that the diameter of Q is equivalent to $\langle \mathrm{dist}(0,Q)\rangle^{\alpha}$.

Corresponding to an α-covering $\mathcal{Q}$, a sequence $\{\rho_Q\}_{Q\in\mathcal{Q}}$ of smooth functions is called a bounded admissible partition of unity of order p (p-BAPU), provided that ρ_Q is supported in Q and

$$\sup_{Q\in\mathcal{Q}} |Q|^{1/(1\wedge p)-1} \|\mathfrak{F}^{-1}\rho_Q\|_{L^{1\wedge p}} < \infty, \quad \sum_{Q\in\mathcal{Q}} \rho_Q(\xi) = 1, \forall\, \xi \in \mathbb{R}^n.$$

[1] $|Q|$ denotes the Lebesgue measure of Q

For $0 \le \alpha \le 1, 0 < p, q \le \infty$ and $s \in \mathbb{R}$, one can introduce Gröbner's α-modulation spaces in the following way

$$M_{p,q}^{s,\alpha}(\mathbb{R}^n) := \left\{ f \in \mathfrak{S}'(\mathbb{R}^n) : \|f\|_{M_{p,q}^{s,\alpha}} := \left(\sum_{Q \in \mathcal{Q}} \langle \mathrm{dist}(0,Q) \rangle^{sq} \|\rho_Q(D) f\|_{L^p}^q \right)^{\frac{1}{q}} < \infty \right\},$$

with some usual modification when $q = \infty$.

Modulation space is special α-modulation space in the case $\alpha = 0$, and Besov space can be regarded as the limit case of α-modulation space when $\alpha \nearrow 1$. Modulation spaces were first introduced by Feichtinger in the study of time-frequency analysis to consider the decay property of a function in both physical and frequency spaces and his original idea is to use the short-time Fourier transform of a tempered distribution equipping with a mixed $L^q(L^p)$-norm to generate $M_{p,q}^s$. Gröchenig's book [12] systematically discussed the theory of time-frequency analysis and modulation spaces.

In the past decade, many works are devoted to the global well-posedness problem for the nonlinear evolution equations with initial data in modulation spaces, particularly, for the NLS (cf. [1, 2, 5, 6, 7, 8, 9, 15, 20, 21, 22, 23])

$$\mathrm{i}\partial_t u + \Delta u = \pm |u|^{2\kappa} u, \quad u(0) = u_0, \tag{1.1}$$

where $\kappa \in \mathbb{N}$, u is a complex valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, Δ denotes the Laplacian on $\mathbb{R}^n$ and u_0 is the initial data at $t = 0$.

It is well known that Cazenave-Weissler [4] obtained the local well-posedness result for NLS in the critical space $H^{\frac{n}{2}-\frac{1}{\kappa}}$, and the local solution is global if the initial data in $\dot{H}^{\frac{n}{2}-\frac{1}{\kappa}}$ is small enough. Namely, we can solve NLS in critical and subcritical Sobolev spaces H^s ($s \ge n/2 - 1/\kappa$). However, up to now there is no systematic method to solve NLS in the supercritical Sobolev spaces H^s ($s < n/2 - 1/\kappa$). Recall the embedding

$$M_{2,1}^{s,\alpha} \subset B_{2,1}^s \subset H^s, \quad M_{2,1}^{s_1,\alpha} \not\subset B_{2,1}^{s_2}, \ s_1 < s_2.$$

If we can solve NLS in α-modulation spaces $M_{2,1}^{s,\alpha}$ for some $s < n/2 - 1/\kappa$, then we obtain the well-posedness of NLS with a class of data out of the critical Sobolev spaces $\dot{H}^{\frac{n}{2}-\frac{1}{\kappa}}$.

§ 2. Definition and basic properties of α-modulation spaces

First, we define α-modulation spaces using a delicate p-BAPU ([3]). Let us denote

$$\ell_{s,\alpha}^q(\mathbb{Z}^n; L^p) = \left\{ \{g_k\}_{k \in \mathbb{Z}^n} : g_k \in \mathfrak{S}'(\mathbb{R}^n), \left\| \langle k \rangle^{\frac{s}{1-\alpha}} \|g_k\|_p \right\|_{\ell^q} < \infty \right\}.$$

If there is no explanation, we will always assume that

$$s \in \mathbb{R}, \quad 0 < p, q \leqslant \infty, \quad 0 \leqslant \alpha < 1.$$

Let us start with the third partition of unity on frequency space for $\alpha \in [0,1)$. We suppose $c < 1$ and $C > 1$ are two positive constants, which relate to the space dimension n, and a Schwartz function sequence $\{\eta_k^\alpha\}_{k\in\mathbb{Z}^n}$ satisfying

$$|\eta_k^\alpha(\xi)| \gtrsim 1, \quad \forall \xi : \left|\xi - \langle k\rangle^{\frac{\alpha}{1-\alpha}} k\right| < c\langle k\rangle^{\frac{\alpha}{1-\alpha}}; \tag{2.1a}$$

$$\text{supp}\eta_k^\alpha \subset \left\{\xi : \left|\xi - \langle k\rangle^{\frac{\alpha}{1-\alpha}} k\right| < C\langle k\rangle^{\frac{\alpha}{1-\alpha}}\right\}; \tag{2.1b}$$

$$\sum\nolimits_{k\in\mathbb{Z}^n} \eta_k^\alpha(\xi) \equiv 1, \quad \forall \xi \in \mathbb{R}^n; \tag{2.1c}$$

$$\langle k\rangle^{\frac{\alpha|\beta|}{1-\alpha}} \left|D^\beta \eta_k^\alpha(\xi)\right| \lesssim 1, \quad \forall \xi \in \mathbb{R}^n. \tag{2.1d}$$

We denote

$$\Upsilon = \left\{\{\eta_k^\alpha\}_{k\in\mathbb{Z}^n} : \{\eta_k^\alpha\}_{k\in\mathbb{Z}^n} \text{ satisfies } (2.1\text{a}) - (2.1\text{d})\right\} \tag{2.2}$$

Corresponding to every sequence $\{\eta_k^\alpha\}_{k\in\mathbb{Z}^n} \in \Upsilon$, one can construct an operator sequence denoted by $\{\Box_k^\alpha\}_{k\in\mathbb{Z}^n}$, and

$$\Box_k^\alpha = \mathfrak{F}^{-1}\eta_k^\alpha \mathfrak{F}. \tag{2.3}$$

Υ is nonempty. Indeed, let ρ be a smooth radial bump function supported in $B(0,2)$, satisfying $\rho(\xi) = 1$ as $|\xi| < 1$, and $\rho(\xi) = 0$ as $|\xi| \geqslant 2$. For any $k \in \mathbb{Z}^n$, we set

$$\rho_k^\alpha(\xi) = \rho\left(\xi - \langle k\rangle^{\frac{\alpha}{1-\alpha}} k \Big/ \langle k\rangle^{\frac{\alpha}{1-\alpha}}\right)$$

and denote

$$\eta_k^\alpha(\xi) = \rho_k^\alpha(\xi)\left(\sum_{l\in\mathbb{Z}^n} \rho_l^\alpha(\xi)\right)^{-1}.$$

It is easy to verify that $\{\eta_k^\alpha\}_{k\in\mathbb{Z}^n}$ satisfies (2.1). This type of decomposition on frequency space is a generalization of the uniform decomposition and the dyadic decomposition. When $0 \leqslant \alpha < 1$, on the basis of this decomposition, we define the α-modulation space by

$$M_{p,q}^{s,\alpha}(\mathbb{R}^n) = \left\{f \in \mathfrak{S}'(\mathbb{R}^n) : \|f\|_{M_{p,q}^{s,\alpha}} := \|\{\Box_k^\alpha f\}_{k\in\mathbb{Z}^n}\|_{\ell_{s,\alpha}^q(\mathbb{Z}^n;L^p)} < \infty\right\}. \tag{2.4}$$

We mention that either $\{\eta_k^\alpha\}_{k\in\mathbb{Z}^n}$ or $\{\rho_k^\alpha\}_{k\in\mathbb{Z}^n}$ forms p-BAPU according to Gröbner's definition, and generates the same α-modulation spaces with equivalent norms.

Strictly speaking, the definition (2.4) dose not cover the case $\alpha = 1$, however, we will denote $M_{p,q}^{s,1} = B_{p,q}^s$ for convenience.

The basic properties of α-modulation spaces include:

Proposition 2.1 (**Completeness**). *$M_{p,q}^{s,\alpha}$ is a quasi-Banach space, and is a Banach space unless $1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$. We have*

$$\mathfrak{S}(\mathbb{R}^n) \subset M_{p,q}^{s,\alpha}(\mathbb{R}^n) \subset \mathfrak{S}'(\mathbb{R}^n). \tag{2.5}$$

Moreover, if $0 < p, q < \infty$, then $\mathfrak{S}(\mathbb{R}^n)$ is dense in $M_{p,q}^{s,\alpha}$.

Proposition 2.2 (Embedding). *Suppose $p_1 \leqslant p_2$, we have*
(i) if $q_1 \leqslant q_2$ and $s_1 \geqslant s_2 + n\alpha\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, then

$$M^{s_1,\alpha}_{p_1,q_1} \subset M^{s_2,\alpha}_{p_2,q_2}; \tag{2.6}$$

(ii) if $q_1 > q_2$ and $s_1 > s_2 + n\alpha\left(\frac{1}{p_1} - \frac{1}{p_2}\right) + n(1-\alpha)\left(\frac{1}{q_2} - \frac{1}{q_1}\right)$, then

$$M^{s_1,\alpha}_{p_1,q_1} \subset M^{s_2,\alpha}_{p_2,q_2}. \tag{2.7}$$

Proposition 2.3. *$M^{s,\alpha}_{2,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$ with equivalent norms.*

It is known that the dual space of Besov space $B^s_{p,q}$ is $B^{-s+n(1/(p\wedge 1)-1)}_{(p\vee 1)^*,(q\vee 1)^*}$ (see [19]) and the dual space of modulation space $M^s_{p,q}$ is $M^{-s}_{(p\vee 1)^*,(q\vee 1)^*}$ (see [21]). The duality for α-modulation spaces is:

Theorem 2.4. *Suppose $0 < p, q < \infty$, then we have*

$$\left(M^{s,\alpha}_{p,q}\right)^* = M^{-s+n\alpha\left(\frac{1}{1\wedge p}-1\right)}_{(1\vee p)^*,(1\vee q)^*}. \tag{2.8}$$

The complex interpolation for Besov spaces has a perfect theory; cf. [19]. Without any essential difference, we imitate the counterpart for the Besov space to construct the complex interpolation for α-modulation spaces, which will be repeatedly used in the following argument.

Theorem 2.5. *Suppose $0 < \theta < 1$ and*

$$s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \tag{2.9}$$

then we have

$$\left(M^{s_0,\alpha}_{p_0,q_0}, M^{s_1,\alpha}_{p_1,q_1}\right)_\theta = M^{s,\alpha}_{p,q}. \tag{2.10}$$

§ 3. Scaling property of α-modulation spaces

Suppose $f \in \mathfrak{S}'(\mathbb{R}^n)$ and $\lambda > 0$, we write $f_\lambda(\cdot) = f(\lambda\cdot)$. For Besov space, it is well known that

$$\|f_\lambda\|_{B^s_{p,q}} \lesssim \lambda^{-\frac{n}{p}}(1 \vee \lambda^s)\|f\|_{B^s_{p,q}}. \tag{3.1}$$

For modulation spaces with $s = 0$ and $1 \le p, q \le \infty$, the dilation property was obtained in Sugimoto and Tomita [17] and they obtained a sharp result:

$$\|f_\lambda\|_{M^0_{p,q}} \lesssim \lambda^{-\frac{n}{p}} \left(1 \vee \lambda^{n\left(\frac{1}{q}-\frac{1}{p}\right)} \vee \lambda^{n\left(\frac{1}{p}+\frac{1}{q}-1\right)}\right) \|f\|_{M^0_{p,q}}. \tag{3.2}$$

In order to study the scaling property of α-modulation spaces, for $0 < p, q \leqslant \infty$ and $(\alpha_1, \alpha_2) \in [0,1] \times [0,1]$, we define

$$R(p,q;\alpha_1,\alpha_2) = 0 \vee \left[n(\alpha_1-\alpha_2)\left(\tfrac{1}{q}-\tfrac{1}{p}\right)\right] \vee \left[n(\alpha_1-\alpha_2)\left(\tfrac{1}{p}+\tfrac{1}{q}-1\right)\right]. \tag{3.3}$$

Then, $\mathbb{R}^2_+$ is divided into three sub-domains in two ways (see Fig.1). One way is, $\mathbb{R}^2_+ = \mathrm{S}_1 \cup \mathrm{S}_2 \cup \mathrm{S}_3$ with

$$\begin{aligned}
\mathrm{S}_1 &= \left\{\left(\tfrac{1}{p},\tfrac{1}{q}\right) \in \mathbb{R}^2_+ : \tfrac{1}{q} \geqslant \tfrac{1}{p}, \tfrac{1}{p} \leqslant \tfrac{1}{2}\right\};\\
\mathrm{S}_2 &= \left\{\left(\tfrac{1}{p},\tfrac{1}{q}\right) \in \mathbb{R}^2_+ : \tfrac{1}{p}+\tfrac{1}{q} \geqslant 1, \tfrac{1}{p} > \tfrac{1}{2}\right\};\\
\mathrm{S}_3 &= \mathbb{R}^2_+ \backslash \{\mathrm{S}_1 \cup \mathrm{S}_2\},
\end{aligned}$$

Another way is, $\mathbb{R}^2_+ = \mathrm{T}_1 \cup \mathrm{T}_2 \cup \mathrm{T}_3$ with

$$\begin{aligned}
\mathrm{T}_1 &= \left\{\left(\tfrac{1}{p},\tfrac{1}{q}\right) \in \mathbb{R}^2_+ : \tfrac{1}{p} \geqslant \tfrac{1}{q}, \tfrac{1}{p} > \tfrac{1}{2}\right\};\\
\mathrm{T}_2 &= \left\{\left(\tfrac{1}{p},\tfrac{1}{q}\right) \in \mathbb{R}^2_+ : \tfrac{1}{p}+\tfrac{1}{q} \leqslant 1, \tfrac{1}{p} \leqslant \tfrac{1}{2}\right\};\\
\mathrm{T}_3 &= \mathbb{R}^2_+ \backslash \{\mathrm{T}_1 \cup \mathrm{T}_2\}.
\end{aligned}$$

If $\alpha_1 \geqslant \alpha_2$, then

$$R(p,q;\alpha_1,\alpha_2) = \begin{cases} n(\alpha_1-\alpha_2)\left(\tfrac{1}{q}-\tfrac{1}{p}\right), & \left(\tfrac{1}{p},\tfrac{1}{q}\right) \in \mathrm{S}_1;\\ n(\alpha_1-\alpha_2)\left(\tfrac{1}{p}+\tfrac{1}{q}-1\right), & \left(\tfrac{1}{p},\tfrac{1}{q}\right) \in \mathrm{S}_2;\\ 0, & \left(\tfrac{1}{p},\tfrac{1}{q}\right) \in \mathrm{S}_3. \end{cases} \tag{3.4}$$

Else if $\alpha_1 < \alpha_2$, then

$$R(p,q;\alpha_1,\alpha_2) = \begin{cases} 0, & \left(\tfrac{1}{p},\tfrac{1}{q}\right) \in \mathrm{T}_3;\\ n(\alpha_1-\alpha_2)\left(\tfrac{1}{p}+\tfrac{1}{q}-1\right), & \left(\tfrac{1}{p},\tfrac{1}{q}\right) \in \mathrm{T}_2;\\ n(\alpha_1-\alpha_2)\left(\tfrac{1}{q}-\tfrac{1}{p}\right), & \left(\tfrac{1}{p},\tfrac{1}{q}\right) \in \mathrm{T}_1. \end{cases} \tag{3.5}$$

Let us write $s_p = n(1/(1 \wedge p) - 1)$ and

$$s_c = \begin{cases} R(p,q;1,\alpha), & \lambda > 1,\\ -R(p,q;\alpha,1), & \lambda \leqslant 1. \end{cases} \tag{3.6}$$

The scaling property for α-modulation spaces can be stated as:

Theorem 3.1. *Let $0 \leqslant \alpha < 1, \lambda > 0$ and $s \neq -s_c$. Then for any $f \in M^{s,\alpha}_{p,q}$, we have*

$$\|f_\lambda\|_{M^{s,\alpha}_{p,q}} \lesssim \lambda^{-\frac{n}{p}} \left[(1 \vee \lambda)^{s_p} \vee \lambda^{s+s_c}\right] \|f\|_{M^{s,\alpha}_{p,q}}. \tag{3.7}$$

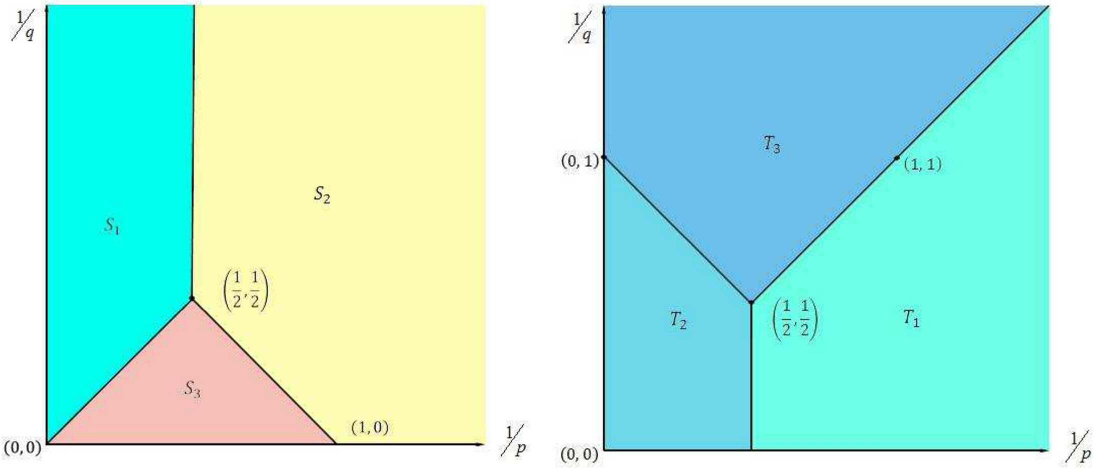

Figure 1. Distribution of s_c. The left-hand side figure is for $\lambda > 1$, the right-hand side figure is for $\lambda \leqslant 1$.

Conversely, if there exists some $F : (0, \infty) \to \mathbb{R}_+$ *such that*

$$\|f_\lambda\|_{M^{s,\alpha}_{p,q}} \lesssim \lambda^{-\frac{n}{p}} F(\lambda) \|f\|_{M^{s,\alpha}_{p,q}}$$

for all $f \in M^{s,\alpha}_{p,q}$, *then* $F(\lambda) \geq (1 \vee \lambda)^{s_p} \vee \lambda^{s+s_c}$.

§ 4. Embedding between different α-modulation spaces and Besov space

As $1 \leq p, q \leq \infty$, some sufficient conditions for the inclusions between modulation and Besov spaces were obtained by Gröbner [11], then Toft [18] improved Gröbner's sufficient conditions, which were proven to be necessary by Sugimoto and Tomita [17]. Their results were generalized to the cases $0 < p, q \leq \infty$ in [21, 22]. Gröbner [11] also considered the inclusions between α_1-modulation and α_2-modulation spaces for $1 \leq p, q \leq \infty$. We improve Gröbner's results in the cases $1 \leq p, q \leq \infty$ and the cases $0 < p, q < 1$ will also be considered.

The embedding between different α-modulation spaces is the following.

Theorem 4.1. *Let* $(\alpha_1, \alpha_2) \in [0, 1) \times [0, 1)$. *Then*

$$M^{s_1,\alpha_1}_{p,q} \subset M^{s_2,\alpha_2}_{p,q} \tag{4.1}$$

if and only if $s \geqslant s_2 + R(p, q; \alpha_1, \alpha_2)$.

The embedding between α-modulation spaces and Besov space is the following.

Theorem 4.2. *Let* $\alpha \in [0, 1)$. *Then* $B^{s_1}_{p,q} \subset M^{s_2,\alpha}_{p,q}$ *if and only if* $s_1 \geqslant s_2 + R(p, q; 1, \alpha)$. *Conversely,* $M^{s_1,\alpha}_{p,q} \subset B^{s_2}_{p,q}$ *if and only if* $s_1 \geqslant s_2 + R(p, q; \alpha, 1)$.

§ 5. Algebraic property of α-modulation spaces

The algebraic property of α-modulation spaces is important, which provides the ways to get the estimate of the nonlinear terms of the partial differential equations.

For Besov space, it is well known that if $s > n/p$, $B^s_{p,q}$ forms a multiplication algebra. But for α-modulation space, the issue is much more complicated. The conditions for which $M^{s,\alpha}_{p,q}$ constitutes a multiplication algebra, are quite different from those of Besov and modulation spaces. Up to now, it is not very clear for us to know the sharp low bound of the index s for which $M^{s,\alpha}_{p,q}$ constitutes a multiplication algebra. When $\big(1/p,\ 1/q\big) \in [0, 1] \times [0, 1]$ (the case of Banach space), the expected ideal critical value is

$$\frac{n\alpha}{p} + n(1-\alpha)\left(1 - \frac{1}{q}\right),$$

but it seems hard to reach it in some area of $(1/p, 1/q)$. We introduce a parameter, denoted by $s_0 = s_0(p, q; \alpha)$, to describe the regularity for which $M^{s,\alpha}_{p,q}$ with $s > s_0$ forms a multiplication algebra. Denote (see Figure 2)

$$\mathrm{D}_1 = \left\{ \left(\tfrac{1}{p}, \tfrac{1}{q}\right) \in \mathbb{R}^2_+ : \tfrac{1}{q} \geqslant \tfrac{2}{p}, \tfrac{1}{p} \leqslant \tfrac{1}{2} \right\}, \quad \mathrm{D}_2 = \mathbb{R}^2_+ \backslash \mathrm{D}_1$$

and

$$s_0 = \begin{cases} \frac{n\alpha}{p} + n(1-\alpha)\big(1 - 1 \wedge \frac{1}{q}\big) + \frac{n\alpha(1-\alpha)}{2-\alpha}\big(\frac{1}{q} - \frac{2}{p}\big), & \big(\frac{1}{p}, \frac{1}{q}\big) \in \mathrm{D}_1; \\ \frac{n\alpha}{p} + n(1-\alpha)\big(1 \vee \frac{1}{p} \vee \frac{1}{q} - \frac{1}{q}\big) + \frac{n\alpha(1-\alpha)}{2-\alpha}\big(1 \vee \frac{1}{p} \vee \frac{1}{q} - 1\big), & \big(\frac{1}{p}, \frac{1}{q}\big) \in \mathrm{D}_2. \end{cases}$$

Theorem 5.1. *If* $s > s_0$, *then* $M^{s,\alpha}_{p,q}$ *is a multiplication algebra, which is equivalent to say that for any* $f, g \in M^{s,\alpha}_{p,q}$, *we have*

$$\|fg\|_{M^{s,\alpha}_{p,q}} \lesssim \|f\|_{M^{s,\alpha}_{p,q}} \|g\|_{M^{s,\alpha}_{p,q}}. \tag{5.1}$$

We sketch the ideas in the proof of the algebraic structure. According to the definition,

$$\|fg\|_{M^{s,\alpha}_{p,q}} = \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{qs/(1-\alpha)} \|\mathfrak{F}^{-1} \eta^\alpha_k \mathfrak{F}(fg)\|^q_{L^p(\mathbb{R}^n)} \right)^{1/q}, \tag{5.2}$$

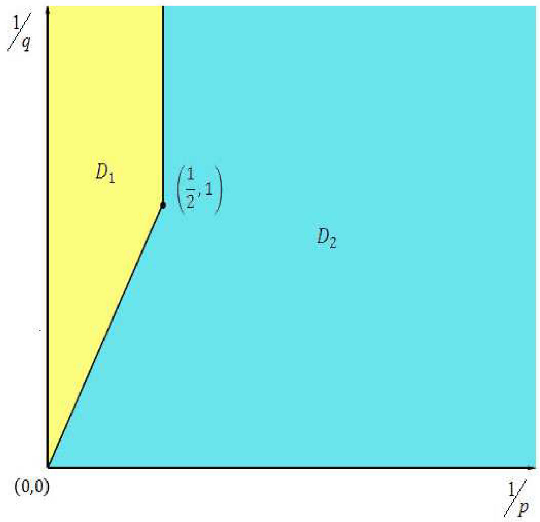

Figure 2. Distribution of s_0

where

$$\mathrm{supp}\eta_k^\alpha \subset \left\{\xi : \left|\xi - \langle k\rangle^{\frac{\alpha}{1-\alpha}} k\right| < C\langle k\rangle^{\frac{\alpha}{1-\alpha}}\right\}.$$

One has that

$$\Box_k^\alpha(fg) = \sum_{k^{(1)}\in\mathbb{Z}^n}\sum_{k^{(2)}\in\mathbb{Z}^n} \Box_k^\alpha(\Box_{k^{(1)}}^\alpha f\ \Box_{k^{(2)}}^\alpha g). \tag{5.3}$$

If we insert (5.3) into (5.2), then we have three summations

$$\sum_{k\in\mathbb{Z}^n},\quad \sum_{k^{(1)}\in\mathbb{Z}^n},\quad \sum_{k^{(2)}\in\mathbb{Z}^n}.$$

So, we must remove the summation on $k \in \mathbb{Z}^n$. The case for modulation spaces ($\alpha = 0$) is easier, since

$$\mathrm{supp}\eta_k^0 \subset \{|\xi - k| \le C\},$$

which leads to

$$|k - k^{(1)} - k^{(2)}| \le C.$$

So, the summation on k is, in fact finite. However, if $0 < \alpha < 1$, from

$$\Box_k^\alpha(\Box_{k^{(1)}}^\alpha f\ \Box_{k^{(2)}}^\alpha g) \ne 0$$

we have

$$\langle k\rangle^{\frac{\alpha}{1-\alpha}}(k_j - C) < \langle k^{(1)}\rangle^{\frac{\alpha}{1-\alpha}}(k_j^{(1)} + C) + \langle k^{(2)}\rangle^{\frac{\alpha}{1-\alpha}}(k_j^{(2)} + C), \tag{5.4}$$

$$\langle k\rangle^{\frac{\alpha}{1-\alpha}}(k_j + C) > \langle k^{(1)}\rangle^{\frac{\alpha}{1-\alpha}}(k_j^{(1)} - C) + \langle k^{(2)}\rangle^{\frac{\alpha}{1-\alpha}}(k_j^{(2)} - C). \tag{5.5}$$

Namely, the summation on k satisfies (5.4) and (5.5). To control the summation on k, one needs to carefully calculate the number of k in (5.4) and (5.5). The condition $s > s_0$ is mainly used for controlling summation on k.

Remark. We know that $M^0_{\infty,1}$ is an algebra. But for $M^{s,\alpha}_{\infty,1}$, we only showed that for $s > \frac{n\alpha(1-\alpha)}{2-\alpha}$, it becomes a Banach algebra. Up to now, we do not know if $s > \frac{n\alpha(1-\alpha)}{2-\alpha}$ is necessary for the algebraic structure of $M^{s,\alpha}_{\infty,1}$. We can show that our results are optimal in some special cases:

Theorem 5.2. *Let $0 \le \alpha < 1$, $(1/p, 1/q) \in D_2$, $p > 1$. If $s < s_0$, then $M^{s,\alpha}_{p,q}$ is not a Banach algebra.*

§ 6. Global well-posedness for NLS in α-modulation spaces

From Theorem 4.2, we see that if $s_1 \ge s_2$, then $M^{s_1,\alpha}_{2,1} \subset B^{s_2}_{2,1}$. However, if $s_1 < s_2$, then $M^{s_1,\alpha}_{2,1} \not\subset B^{s_2}_{2,1}$. In fact, we define f by

$$\widehat{f} = \sum_{|k|\gg 1} \langle k\rangle^{-n-\frac{s_1+s_2}{2(1-\alpha)}} \chi_{B(\langle k\rangle^{\frac{\alpha}{1-\alpha}}k,1)}.$$

Direct calculation shows that $\|f\|_{M^{s_1,\alpha}_{2,1}} \ll 1$ but $\|f\|_{B^{s_2}_{2,1}} = \infty$ (which implies $\|f\|_{H^{s_2}} = \infty$). So, for any $s < n/2 - 1/\kappa$, there exist a class of data, which are small in $M^{s,\alpha}_{2,1}$ but can be arbitrarily large in $H^{n/2-1/\kappa}$.

Now we state a global well posedness result for the NLS in $M^{s,\alpha}_{2,1}$. Let $\theta \in [0,1]$ be a parameter. We set $p_1, q_1; p_2, q_2$ by:

$$\begin{cases} \frac{1}{q_1} = \frac{1-\theta}{\infty} + \frac{\theta}{2\kappa+2} \\ \frac{1}{p_1} = \frac{1-\theta}{2} + \frac{\theta}{2\kappa+2} \end{cases}, \quad \begin{cases} \frac{1}{q_2} = \frac{1-\theta}{2\kappa} + \frac{\theta}{2\kappa+2} \\ \frac{1}{p_2} = \frac{1-\theta}{\infty} + \frac{\theta}{2\kappa+2} \end{cases}. \tag{6.1}$$

For $n \ge 1$, we define θ_1, θ_2 and θ_3 as

$$\begin{aligned} &[\kappa+1+\alpha(n\kappa-2)]\theta_1 = \kappa+1, \\ &[\kappa+1+\alpha(n\kappa-2)]\theta_2 = \frac{2(\kappa+1)}{1-\alpha}\left[(1-\alpha)\left(\frac{n\alpha}{2}+\frac{\alpha}{\kappa}+\frac{1}{2}\right)\right. \\ &\qquad\qquad \left. -(2\kappa+1-\alpha)\left(s-\frac{n\alpha}{2}+\frac{\alpha}{\kappa}\right)\right], \\ &[\kappa+1+\alpha(n\kappa-2)]\theta_3 = \frac{2\kappa(\kappa+1)}{(1+\alpha)\kappa-1}\left[(2-\alpha)\left(s-\frac{n\alpha}{2}+\frac{\alpha}{\kappa}\right)\right. \\ &\qquad\qquad \left. +\frac{(1+\alpha)\kappa-1-\alpha(1-\alpha)(n\kappa+2)}{2\kappa}\right]; \end{aligned} \tag{6.2}$$

and for $n = 1$, we additionally define

$$[\kappa + 1 + \alpha(\kappa - 2)]\theta_4 = (2\kappa + 2)\left(\alpha - s - \frac{1}{2}\right). \tag{6.3}$$

We assume that θ satisfies

$$0 \vee \theta_2 \le \theta \le \theta_3 \wedge \theta_1, \quad n \ge 2, \tag{6.4}$$

$$0 \vee \theta_2 \vee \theta_4 \le \theta \le \theta_3 \wedge \theta_1, \ \theta \ne \theta_4, \quad n = 1. \tag{6.5}$$

The components to construct the compound function spaces:

$$\|u\|_{p_1,q_1;1} = \sum_{i=1}^{n} \sum_{k\in\mathbb{Z}_i^n, |k|\gg 1} \langle k\rangle^{\frac{1}{1-\alpha}\left(s + \frac{1-\theta}{2} - \theta\alpha\frac{n\kappa-2}{2\kappa+2}\right)} \|\Box_k^\alpha u\|_{L^{q_1}_{x_i} L^{p_1}_{(x_j)_{j\ne i}, t}},$$

$$\|u\|_{p_2,q_2;1} = \sum_{i=1}^{n} \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{1}{1-\alpha}\left[s - (1-\theta)\left(\frac{n\alpha}{2} - \frac{\alpha}{\kappa} + \frac{1}{2\kappa}\right) - \theta\alpha\frac{n\kappa-2}{2\kappa+2}\right]} \|\Box_k^\alpha u\|_{L^{q_2}_{x_i} L^{p_2}_{(x_j)_{j\ne i}, t}},$$

$$\|u\|_{p_3,q_3;1} = \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{s}{1-\alpha} - \frac{\alpha}{1-\alpha}\frac{n\kappa-2}{2\kappa+2}} \|\Box_k^\alpha u\|_{L^{2\kappa+2}_{t,x}}.$$

The working space X is defined as:

$$X = \{u \in \mathfrak{S}'(\mathbb{R}^n) : \|u\|_X := \|u\|_{\cap_{i=1}^3 (p_i, q_i; 1)} < \infty\}.$$

We set

$$s_c = \frac{n\alpha}{2} - \frac{\alpha}{\kappa} + \frac{\alpha(1-\alpha)(n\kappa+2)}{2\kappa[(1+\alpha)\kappa + 1 - \alpha]}. \tag{6.6}$$

Theorem 6.1. *Let $n\kappa > 2$, $0 \le \alpha < 1$ and θ be as in (6.4) and (6.5). Suppose $u_0 \in M^{s,\alpha}_{2,1}$ with $s \ge s_c$ and $\|u_0\|_{M^{s,\alpha}_{2,1}} \le \delta$ for some small positive number δ. Then NLS (1.1) has a unique solution*

$$u \in C(\mathbb{R}, M^{s,\alpha}_{2,1}) \cap X,$$

and $\|u\|_X \lesssim \delta$.

Remark. Noticing that $s_c = 0$ for $\alpha = 0$ and $s_c = n/2 - 1/\kappa$ for $\alpha = 1$, which are the critical indices for NLS in modulation and Sobolev spaces, respectively. So, Theorem 6.1 is optimal in the end point cases $\alpha = 0, 1$.

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